

# Swinging over the water hole



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## Abstract

A child takes a running start, grabs hold of the free-hanging end of a rope at the edge of a water hole at  $x=0$ , swings upward, and lets go of the rope at some point, flying freely through the air until he lands in the water at  $x=R$ . For small initial speeds, the maximum range  $R$  is obtained by releasing the rope just short of its turning point and dropping almost straight down into the water from rest. On the other hand, as a child's running speed gets larger and larger (compared to the square root of the product of the length of the rope and earth's gravitational field), the rope should be released at an angle of about  $\pi/4$  to maximize the range. That is, the optimum trajectory of a child is dominated by pendulum motion at low running speeds and by projectile motion at high initial speeds.

**Keywords:** Pendulum, projectile, maximum range, numerical solution.

## Resumen

Un niño se pone en marcha, agarra el extremo libre colgando de una cuerda en el borde de un agujero de agua en  $x=0$ , oscilando hacia arriba, y suelta la cuerda en algunos momentos, volando libremente por el aire hasta que se sumerge en el agua en  $x=R$ . Para pequeñas velocidades iniciales, el máximo rango de  $R$  se obtiene mediante la liberación de la cuerda justo por debajo de su punto de inflexión y cayendo casi en línea recta hacia abajo en el agua desde el reposo. Por otra parte, como la velocidad de carrera del niño se hace más y más grande (en comparación con la raíz cuadrada del producto de la longitud de la cuerda y el campo gravitacional de la Tierra), la cuerda debe ser liberada en un ángulo de alrededor de  $\pi/4$  para maximizar el rango. Es decir, la trayectoria óptima de un niño está dominada por el movimiento del péndulo a bajas velocidades de carrera y por el movimiento de proyectiles a altas velocidades iniciales.

**Palabras clave:** Péndulo, proyectil, alcance máximo, solución numérica.

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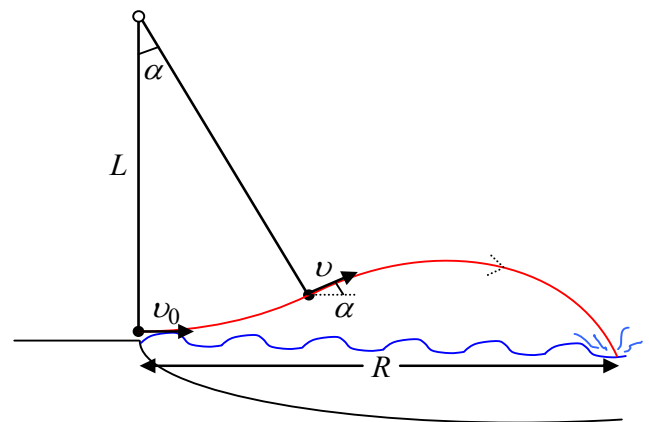
## I. PROBLEM STATEMENT

Suppose some children can individually swing on the end of a long, inextensible rope that just skims the ground at the edge of a pond, as sketched in Fig. 1. Each child takes a running start and jumps onto the rope of length  $L$  with the same initial speed  $v_0$ . At some point during the subsequent upswing, a child lets go of the rope (say when he is at angle  $\alpha$  relative to the vertical) and flies through the air in a parabolic trajectory (neglecting air drag), landing in the water a horizontal distance  $R$  away from the starting point of the swing. The children decide to have a competition and see who can maximize that horizontal range. At what angle  $\alpha_{\max}$  should a child let go of the rope to do so?

## II. SOLUTION

If a child does not let go of the rope, he will swing beyond a

final angle<sup>1</sup> of  $\alpha_f = \pi/2$  if  $v_0^2 > 2gL$ , where  $g$  is earth's



**FIGURE 1.** Trajectory of a child swinging on the end of a rope and then flying through the air into a water hole.

<sup>1</sup>All angles in this paper are in radians.

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surface gravitational field. On the other hand, if  $v_0^2 \leq 2gL$  then he will smoothly swing up to (and back down from) an angle of

$$\alpha_f = \cos^{-1}(1 - v_0^2 / 2gL). \quad (1)$$

For example, if  $v_0 = 7.5$  m/s,  $g = 9.8$  m/s<sup>2</sup>, and  $L = 10$  m, then  $\alpha_f = \pi/4$ . To maximize the range, the rope must be released on the upswing at an angle between 0 and  $\alpha_f$ . As a child lets go of the rope at angle  $\alpha$ , equating the gain in gravitational potential energy to the loss in kinetic energy implies that he will be traveling with velocity

$$v = \sqrt{v_0^2 - 2gy}, \quad (2)$$

at angle  $\alpha$  relative to the horizontal and at a height of

$$y = L(1 - \cos \alpha), \quad (3)$$

above the water. Basic kinematics then implies that his flight time through the air will be

$$t = \frac{v \sin \alpha}{g} + \sqrt{\left(\frac{v \sin \alpha}{g}\right)^2 + \frac{2y}{g}}, \quad (4)$$

where the positive solution of the quadratic formula was selected. His total horizontal range will be

$$R = L \sin \alpha + vt \cos \alpha. \quad (5)$$

This range is independent of a child's mass, because neither pendulum nor projectile motion depends on the mass of an object.

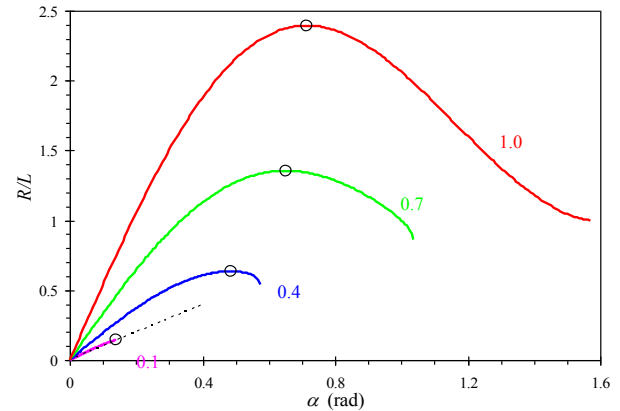
The variables  $\alpha$ ,  $v_0$ , and  $R$  can be recast in dimensionless form as  $a \equiv \cos \alpha$ ,  $w \equiv v_0 / \sqrt{2gL}$ , and  $r \equiv R / L$ . Then Eqs. (2) through (5) can be combined as

$$r = \sqrt{1 - a^2} + 2a\sqrt{(1 - a)(w^2 - 1 + a)} \times \left[ \sqrt{(1 + a)(w^2 - 1 + a)} + \sqrt{(1 + a)w^2 + a^2} \right]. \quad (6)$$

Equation (6) is plotted in Fig. 2 as a function of  $\alpha$  for 4 different values of  $w$ . As these curves suggest, there is a single well-defined maximum for all  $w > 0$ . In particular, with a bit of work one can prove both that  $dr/d\alpha > 0$  as  $\alpha \rightarrow 0$  and that  $dr/d\alpha < 0$  as  $\alpha \rightarrow \alpha_f$ , thus establishing that  $0 < \alpha_{\max} < \alpha_f$ .

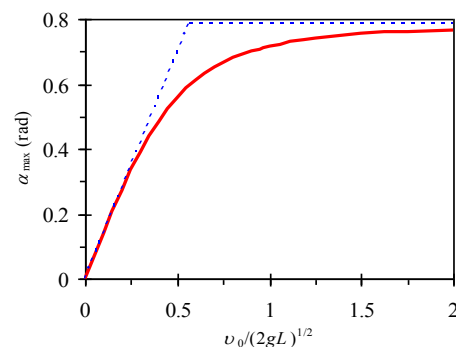
A numerical value of  $\alpha_{\max}$  can be obtained by differentiating Eq. (6) with respect to  $a$  for fixed  $w$  and setting the result equal to zero. These operations were performed in Mathematica™ to obtain the optimal release

angles [1] graphed in Fig. 3. For large speeds  $v_0$ , maximum range is obtained for an angle of  $\pi/4$ , just as for a surface-to-surface projectile, because the release height  $y$  can be neglected at high speeds. On the other hand, if  $w \ll 1$  then Eq. (1) implies that  $\alpha_f \ll 1$  and hence that  $\alpha \ll 1$ . Expanding Eq. (6) up to first powers in  $\alpha$  and  $w$  one obtains



**FIGURE 2.** Range as a function of release angle (varying from 0 to  $\alpha_f$ ) for values of the normalized initial speed indicated by the numbers beside each curve. The circled point at the peak of each curve identifies the coordinates of  $(\alpha_{\max}, r_{\max})$ . The straight dashed line has unit slope.

$r \approx \alpha$ . This result is plotted as the dashed line in Fig. 2 and it approximates well<sup>2</sup> the curve graphed for  $w = 0.1$ . It follows that  $\alpha_{\max} \approx \alpha_f$  for  $w \ll 1$ . Intuitively it makes sense that a child should let go of the rope near the turning point, because for small angles he is essentially traveling horizontally out over the water. But for small  $w$ , Eq. (1) implies that  $\alpha_f \approx \sqrt{2}w$ . Thus  $\alpha_{\max} \approx \sqrt{2}w$  for  $w \ll 1$ , plotted as a diagonal line in Fig. 3.



**FIGURE 3.** Release angle that maximizes the range as a function of the normalized initial speed. The diagonal dashed line extending away from the origin has a slope of  $\sqrt{2}$ , and the joining horizontal dashed line indicates the asymptotic value of  $\pi/4$  that  $\alpha_{\max}$  approaches for large  $v_0$ .

<sup>2</sup>Although it is not evident on the scale of Fig. 2, the curve for  $w = 0.1$  has a slight downward hook at its upper end so that  $\alpha_{\max}$  is smaller than  $\alpha_f$  as is the case for all positive values of  $w$ .

### III. CLOSING REMARKS

This problem could be presented as a case study in an introductory physics class, providing students an opportunity to combine ideas of projectile and pendulum motion. With suitable guidance, they should be able to understand the low- and high-speed limiting expressions for the optimal release angle, graphed as the dashed lines in Fig. 3. Even if they cannot derive or numerically solve Eq. (6), an interesting experimental project [2] would be to measure the range of a pendulum bob released at different angles of upswing for various speeds at the lowest point, to reproduce Fig. 2. A convenient way to vary that speed is to release the bob from different initial angles  $\alpha_0$  (or equivalently from different starting heights  $y_0 = L - L \cos \alpha_0$ ) because

$$w = \sqrt{1 - \cos \alpha_0} . \quad (7)$$

One could use a metal bob that is electromagnetically held

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onto the end of a hinged rod<sup>3</sup> whose swing passes through a photogate that trips off the electromagnet. One practical application of these measurements is they indicate the minimum distance that spectators must keep back from a pendulum to be safe in case the bob happens to come loose. Another is they suggest how large a sandpit should be built around a playground swing, given that children sometimes like to jump off while in motion.

### REFERENCES

- [1] Bittel, D., *Maximizing the range of a projectile launched by a simple pendulum*, Phys. Teach. **43**, 98–100 (2005). Equation (8) in Bittel's paper implies that  $\cos^3 \alpha_{\max} + (2w^2 - 1)\cos^2 \alpha_{\max} = w^2$ , which can be analytically solved for  $\alpha_{\max}$  using Cardano's formula; the results agree with Fig. 3.
- [2] Blisard, T. J. and Duursema, C. H., *A demonstration of the transformation of mechanical energy for student computation*, Am. J. Phys. **20**, 559–561 (1952).

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<sup>3</sup>An advantage of a rod over a string is that initial angles  $\alpha_0$  up to  $\pi$  are possible.